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CONVERGENCE PROPERTIES OF SPLINE FUNCTIONS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled CONVERGENCE PROPERTIES OF SPLINE FUNCTIONS submitted by YANG MAO in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The purpose of this thesis is to investigate the convergence properties of cubic spline functions and quintic spline functions. The basic definitions are introduced in Chapter I and the convergence properties of cubic splines are described in Chapter II. The results in this chapter are not new. Their inclusion is justified by the necessity of enumerating the known facts in this field. The results are due to Ahlberg, Nilson, Walsh, Meir, Sharma, Hall, Marsden and others.

Chapter III of this work develops a sequence of convergence properties for quintic splines, which extend similar results concerning cubic splines. The main results in this chapter which are proved by the author are Theorems III.1.5 to III.1.7, Theorem III.3.1 and Theorem III.3.2. The statements of these theorems are as follows:

Let $\Delta : 0 = x_o < x_1 < \dots < x_{n-1} < x_n = 1$ be a partition of $[0,1]$.

Theorems III.1.5 to III.1.7. (With $p = 1,2,3,$ respectively.)

Suppose $f(x) \in C^p[a,b]$ and has period 1. If $s(x) \in C^4[a,b]$ is the periodic <u>quintic</u> spline interpolating $f(x)$ on Δ (with $x_i = \frac{i}{n}$, $i = 0,\dots,n$). Then

$$|f^{(r)}(x)-s^{(r)}(x)| \leq K_p \, n^{r-p}\omega(f^{(r)};\tfrac{1}{n}) , \quad (r = 0,\dots,p) .$$

Theorem III.3.1

Suppose $f(x) \varepsilon C^2[0,1]$ and has period 1. Let $s(x) \varepsilon C^2[0,1]$ be the periodic quintic spline interpolating $f(x)$ on Δ (with $x_i = \frac{i}{n}$, $i = 0,1,\ldots,n$) and satisfying $f^{(r)}(\frac{i+\lambda-1}{n}) = s^{(r)}(\frac{i+\lambda-1}{n})$, $r = 1,2$. $0 < \lambda < 1$. Then

$$|f^{(r)}(x)-s^{(r)}(x)| \leq K(\lambda)n^{r-2}\omega(f'';\tfrac{1}{n}) \;, \qquad (r = 0,1,2) \;.$$

Theorem III.3.2

Suppose $f(x) \varepsilon C^1(-\infty,\infty)$ and has period 1. Let $s(x) \varepsilon C^1(-\infty,\infty)$ be the quintic spline on Δ with period 1, and with $f^{(r)}(\xi_i) = s^{(r)}(\xi_i)$, $f^{(r)}(\eta_i) = s^{(r)}(\eta_i)$, $r = 0,1$, where $\xi_i = mx_i + (1-m)x_{i-1}$, $\eta_i = \ell x_i + (1-\ell)x_{i-1}$; $0 < m < \frac{1}{2} < \ell < 1$, $m + \ell = 1$. Then for all x we have

$$|f^{(r)}(x)-s^{(r)}(x)| \leq C(\ell,m)||\Delta||^{1-r}\omega(f';||\Delta||) \;, \qquad (r = 0,1) \;,$$

where $||\Delta|| = \max_i |x_i-x_{i-1}|$.

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CHAPTER ONE

INTRODUCTION

§I.1 Spline Functions on a Finite Interval

Let $\Delta : a = x_o < x_1 < \ldots < x_n < x_{n+1} = b$ be a sequence of real numbers. A spline function $s(x)$ of degree k with joints $x_1, x_2, \ldots, x_n$ is a real-valued function having the following properties:

(a) $s(x) \in C^{k-1}[a,b]$;

(b) $s(x)$ is a polynomial of degree not greater than k in each interval $[x_{i-1}, x_i]$ where $i = 1,2,\ldots,n+1$.

Thus a spline function is a piecewise polynomial satisfying certain conditions regarding continuity of the function and its derivatives. When $k = 0$ condition (a) is omitted and a spline function of degree 0 is a step function. A spline function of degree one is a polygon.

We denote by $S_k(x_1, x_2, \ldots, x_n)$ the class of all splines of order k having joints $x_1, x_2, \ldots, x_n$. Since in very special cases $s(x)$ might be given by a single polynomial on the interval $a \leq x \leq b$. This class of function $S_k(x_1, x_2, \ldots, x_n)$ includes the class of all polynomials of order not greater than k.

For $k > 0$, a spline function of degree k could equally well be defined as a function in C^{k-1} whose kth derivative is a step function. Even more concisely, a spline function of degree k is any k^{th} order indefinite integral of a step function. For convenience we shall call a spline function of degree three a cubic spline and a spline function of degree five a quintic spline.

Spline functions of odd degree with suitable boundary conditions have the property of minimizing the integral $\int [s^m(x)]^2 dx$ for given values of $s(x_i)$. Thus, cubic spline functions represent logical genralization of piecewise linear functions, as a scheme for piecewise polynomial interpolation. For given $y_i = s(x_i)$ and endslopes $y'_o = s'(x_o)$, $y'_{n+1} = s'(x_{n+1})$, they are easily calculated as follows.

The condition that $s(x) \varepsilon C^2[a,b]$, for cubic $s(x)$ in the intervals $[x_{i-1},x_i]$ and $[x_i,x_{i+1}]$, is quivalent to the following linear equations:

$$h_{i+1}s'(x_{i-1}) + 2(h_{i+1}+h_i)s'(x_i) + h_i s'(x_{i+1})$$
$$= 3[\Delta y_i \frac{h_{i+1}}{h_i} + \Delta y_{i+1} \frac{h_i}{h_{i+1}}] \tag{1.1.1}$$

where $h_i = x_i - x_{i-1}$ and $\Delta y_i = s(x_i) - s(x_{i-1})$. The resulting system of $n-1$ equations is linearly independent, tridiagonal and diagonally dominant. Hence it can be solved and the values $s'(x_i)$ can be obtained. On the basis of $s(x_i)$ and $s'(x_i)$, $[i = 0,1,\ldots,n+1]$ one easily computes $s(x)$ in each interval $[x_{i-1},x_i]$ by Hermite interpolation.

The preceding method can be adapted to cover also the "free endpoint" conditions $s''(x_o) = s''(x_{n+1}) = 0$. In this case, one must supplement (1.1.1) by

$$2s'(x_o) + s'(x_1) = 3 \frac{\Delta y_1}{h_1} \quad ,$$

$$s'(x_n) + 2s'(x_{n+1}) = 3 \frac{\Delta y_{n+1}}{h_{n+1}} \quad .$$

§I.2 Developments in the Theory of Convergence

We consider a fixed interval $a \leq x \leq b$, and subdivide it by a sequence of points (we call it a mesh) corresponding to the locations of the joints

$$\Delta : a = x_o < x_1 < \ldots < x_N = b \tag{1.2.1}$$

An associated set of ordinates is prescribed:

$$Y : y_o, y_1, \ldots, y_N \; .$$

For any fixed integer $k \leq N$, we seek a spline function $s_\Delta(Y;x)$ of degree k, which interpolates to the values of y_i at the joints of Δ. The spline is said to be periodic of period $(b-a)$ if the condition

$$s_\Delta^{(r)}(Y;a+) = s_\Delta^{(r)}(Y;b-) \qquad (r = 0,1,\ldots,k-1)$$

is satisfied.

It is well known [5] that the cubic spline with perscribed ordinates Y at mesh points (1.2.1) always exists and is unique. If (1.2.1) is a uniform mesh, then periodic polynomial splines of degree k which interpolate to Y at (1.2.1) will exist if one of the following conditions is satisfied:

(a) k is odd and $k \leq N$ (N is the number of mesh intervals)

(b) k is even, $k \leq N$ and N is odd.

When we assume that $f(x)$ is a periodic function (or, that $f(x)$ satisfies certain given end conditions) and (1.2.1) is the subdivision of $[a,b]$ $(N \geq 2k-1)$, then in [4] it was proved that

there exists exactly one spline function $s_\Delta(f;x)$ of degree $2k-1$, interpolating to $f(x)$ at the joints x_i, $(i = 1,2,\ldots,N)$.

It is of interest to investigate the convergence of the spline approximations $s_\Delta^{(r)}(f;x)$ to the approximated functions $f^{(r)}(x)$ as the mesh norm $||\Delta|| = \max_i |x_i - x_{i-1}|$ approaches zero. The first results were obtained by Walsh, Ahlberg, and Nilson [28], for cubic splines. Under the assumption that $f(x) \varepsilon C^2[a,b]$, it was shown that if $s_\Delta(f;x)$ is the interpolatory spline to $f(x)$ at the mesh points, then $s_\Delta^{(r)}(f;x)$ converges uniformly to $f^{(r)}(x)$ for $r = 0,1$. Later, this result was extended by Sharma and Meir ([25] and [26]), who proved, that if $f(x) \varepsilon C^2[a,b]$, then $s_\Delta^{(r)}(f;x)$ converges uniformly to $f^{(r)}(x)$ for $r = 0,1,2$.

Birkhoff and deBoor [8] have shown that if $f(x) \varepsilon C^4[a,b]$ then

$$|f^{(r)}(x) - s_\Delta^{(r)}(f;x)| < K\,||\Delta||^{4-r}, \quad (r = 0,1,\ldots,3) \tag{1.2.3}$$

provided the ratio $R_\Delta = \max_i \cdot \frac{||\Delta||}{h_i}$ is bounded. Under weaker restrictions imposed on $f(x)$ (such as $f(x) \varepsilon C[a,b]$ or $f(x) \varepsilon C^1[a,b]$) appropriate convergence properties have been obtained by Ahlberg, Nilson and Walsh [5]. In addition, the convergence of polynomial splines of odd degree has been investigated by Ahlberg, Nilson and Walsh [4], Schoenberg [21] and Ziegler [29]. Multidimensional splines were the subject of Ahlberg, Nilson and Walsh [5].

Many of the above mentioned convergence results depend on the fine structure of the linear system of equations defining the spline. On the other hand, a number of convergence results can be

established without appeal to the defining equations. In particular, for polynomial splines of degree $\leq 2m-1$ this can be done with respect to the convergence of derivatives of order $\leq m-1$. Moreover, with the aid of the integral relation

$$\int_b^a [f^{(m)}(x)-s_\Delta^{(m)}(f;x)]^2 dx = \int_a^b [f(x)-s_\Delta(f;x)]f^{(2m)}(x)dx \ , \qquad (1.2.4)$$

(see Ahlberg, Nilson and Walsh [4]), convergence of the derivatives of order $\leq 2m-2$ can be established. With the help of (1.2.4) Ahlberg, Nilson and Walsh [5] obtained the following result:

$$|f^{(r)}(x)-s^{(r)}(f;x)| \leq K \,||\Delta||^{2m-r-1}, \qquad (r = 0,1,\ldots,2m-1) \qquad (1.2.5)$$

provided R_Δ is bounded. For cubic splines, this result is weaker than (1.2.3). Whether $2m-r-1$ can be replaced, in general, by $2m-r$ is an open question.

CHAPTER TWO

CUBIC SPLINES

§II.1 Fundamental Convergence Theorems

Let

$$\Delta_k : a = x_{k,o} < x_{k,1} < \ldots < x_{k,N_k-1} < x_{k,N_k} = b \qquad (2.1.1)$$

be a sequence of meshes on the closed interval $[a,b]$. We denote

$$h_{k,i} = x_{k,i} - x_{k,i-1}\,, \qquad ||\Delta_k|| = \max_i h_{k,i}$$

$$R_{\Delta_k} = \max_i\, [||\Delta_k||/h_{k,i}]\ .$$

The first main result of this section is due to Ahlberg and Nilson [5].

Theorem II.1.1

Suppose $f(x) \varepsilon C[a,b]$ and is periodic with period $b-a$. Let $S_{\Delta_k} \varepsilon C[a,b]$ be the periodic spline interpolating to $f(x)$ on the joints (2.1.1) with $||\Delta_k|| \to 0$ as $k \to \infty$ and with $R_{\Delta_k} \leq \beta < \infty$. Then for all $x \varepsilon [a,b]$ we have

$$|f(x) - S_{\Delta_k}(x)| \leq K\cdot\omega(f;||\Delta_k||) \qquad (2.1.2)$$

where $K = \frac{3}{2}(\frac{1}{3^{5/2}}\ \beta^2 + 1)$, and $\omega(f;||\Delta_k||)$ is the modulus of continuity of $f(x)$.

If we assume that $f'(x)$ is continuous on $[a,b]$, then the spline and its derivative converge. Furthermore, it is no longer

required that the condition $R_{\Delta_k} \leq \beta < \infty$ be satisfied. The following theorem is due to Cheney and Schurer [10].

Theorem II.1.2

Suppose $f(x) \in C'[a,b]$ and is periodic with period $b-a$. Let $S_{\Delta_k} \in C'[a,b]$ be the periodic spline interpolating to $f(x)$ at the joints (2.1.1) with $||\Delta_k|| \to 0$ as $k \to \infty$. Then for all $x \in [a,b]$ we have

$$|f^{(r)}(x) - S^{(r)}_{\Delta_k}(x)| \leq \frac{17}{2} ||\Delta_k||^{1-r} \omega(f';||\Delta_k||) \quad ,$$

$$(r = 0,1) \ . \qquad (2.1.3)$$

For the case $f''(x) \in C[a,b]$, Sharma and Meir [26] proved a more general result:

Theorem II.1.3

Suppose $f(x) \in C^2[a,b]$ and is periodic with period $b-a$. Let $S_{\Delta_k}(x) \in C^2[a,b]$ be the periodic spline interpolating to $f(x)$ at the joint (2.1.1) with $||\Delta_k|| \to 0$ as $k \to \infty$. Then for all $x \in [a,b]$ we have

$$|f^{(r)}(x) - S^{(r)}_{\Delta_k}(x)| \leq 5 ||\Delta_k||^{2-r} \omega(f''; ||\Delta_k||) \quad ,$$

$$(r = 0,1,2) \ . \qquad (2.1.4)$$

The estimate (2.1.4) remains valid if the periodicity of $f(x)$ and $S_{\Delta_k}(x)$ is replaced by the requirement

$$f'(a) = S'_{\Delta_k}(a) \qquad \text{and} \qquad f'(b) = S'_{\Delta_k}(b) \ .$$

The convergence properties become even more striking as we increase the smoothness of $f(x)$. Birkhoff and deBoor [8] show that if $f'''(x)$ is absolutely continuous in $[a,b]$ then $S'''_{\Delta_k}(x)$ converges uniformily to $f'''(x)$ provided the mesh condition $R_{\Delta_k} \leq \beta < \infty$ is satisfied. Ahlberg, Nilson and Walsh obtain a slightly stronger form of this theorem.

Theorem II.1.4

Let $f(x) \varepsilon C^3[a,b]$, f be periodic. Let $S_{\Delta_k} \varepsilon C^2[a,b]$ be the periodic spline interpolating to $f(x)$ at the joints (2.1.1) with $||\Delta_k|| \to 0$ as $k \to \infty$ and with $R_{\Delta_k} \leq \beta < \infty$. Then for all $x \varepsilon [a,b]$, we have

$$|f^{(r)}(x) - S^{(r)}_{\Delta_k}(x)| \leq C\ ||\Delta||^{3-r}\ \omega(f''';||\Delta||)\ ,$$

$$(r = 0,1,2,3) \qquad (2.1.5)$$

where $C = 1 + \beta(1 + \beta)^2$

For $f(x) \varepsilon C^4[a,b]$. Birkhoff and deBoor [8] have shown that if $f'(a) = S'_{\Delta_k}(b)$, $f'(b) = S_{\Delta_k}(b)$ then $S^{(r)}_{\Delta_k}$ converges uniformly to $f^{(r)}(x)$ for $r = 1,2,3$ provided the mesh condition $R_{\Delta_k} \leq \beta < \infty$ is satisfied. Here we present a slightly stronger result of C.A. Hall [15].

Theorem II.1.5

Let $f(x) \varepsilon C^4[a,b]$. Let $S_{\Delta_k} \varepsilon C^2[a,b]$ be the cubic spline interpolating to $f(x)$ at the joints (2.1.1) with $||\Delta_k|| \to 0$ as $k \to \infty$. Suppose $R_{\Delta_k} \leq \beta < \infty$ and $S'_{\Delta_k}(x_{k,i}) = f'(x_{k,i})$, $i = 0,N_k$. Then for all $x \varepsilon [a,b]$ we have

$$|f^{(r)}(x) - S^{(r)}_{\Delta_k}(x)| \leq \varepsilon_r ||f^{(4)}|| \; ||\Delta_k||^{4-r} \quad ,$$

$$(r = 0,1,2,3) \qquad (2.1.6)$$

where

$$||f^{(4)}|| = \max \{|f^{(4)}(x)| : x \varepsilon [a,b]\} \quad .$$

$$\varepsilon_o = \frac{5}{384} , \qquad \varepsilon_1 = \frac{1}{216} (9 + \sqrt{3}) \quad ,$$

$$\varepsilon_2 = \frac{1}{12} (3\beta + 1) \qquad \text{and} \qquad \varepsilon_3 = \frac{1}{2} (1 + \beta^2) \quad .$$

§II.2 Additional Convergence Theorems.

Throughout this section we shall be concerned with some results proved by Ahlberg, Nilson and Walsh [5] which extend the theorems in §II.1 from the periodic case to the case of other end conditions.

We classify the end conditions in terms of the $M_{k,j} = S'_{\Delta_k}(f;x_j)$ as follows:

at $x = a$

$$\text{(i)} \quad 2M_{k,o} + M_{k,1} = \frac{6}{h_{k,1}} \left(\frac{f(x_{k,1}) - f(a)}{h_{k,1}} - f'(a)\right) \quad ;$$

$$\text{(ii)} \quad 2M_{k,o} = 2f''(a) \quad ; \qquad (2.2.1)$$

$$\text{(iii)} \quad 2M_{k,o} + \lambda_{k,o} M_{k,1} = d_{k,o} \quad .$$

at $x = b$

$$\text{(i)} \quad M_{k,N_k-1} + 2M_{k,N_k} = \frac{6}{h_{k,N_k}} \left(f'(b) - \frac{f(b) - f(x_{k,N_k-1})}{h_{k,N_k}}\right) ;$$

$$\text{(ii)} \quad 2M_{k,N_k} = 2f''(b) ; \qquad (2.2.2)$$

$$\text{(iii)} \quad U_{k,N_k} M_{k,N_k-1} + 2M_{k,N_k} = d_{k,N_k} .$$

In terms of the $m_{k,j} = S'_{\Delta_k}(f;x_j)$:

at $x = a$

$$\text{(i)} \quad 2m_{k,o} = 2f'(a) ;$$

$$\text{(ii)} \quad 2m_{k,o} + m_{k,1} = 3 \frac{f(x_{k,1}) - f(a)}{h_{k,1}} - \frac{h_{k,1}}{2} f''(a) ; \qquad (2.2.3)$$

$$\text{(iii)} \quad 2m_{k,o} + U_{k,o} m_{k,1} = C_{k,o} .$$

at $x = b$

$$\text{(i)} \quad 2m_{k,N_k} = 2f'(b) ;$$

$$\text{(ii)} \quad m_{k,N_k-1} + 2m_{N_k} = 3 \frac{f(b) - f(x_{k,N_k-1})}{h_{N_k}} + \frac{h_{k,N_k}}{2} f''(b) ; \qquad (2.2.4)$$

$$\text{(iii)} \quad \lambda_{k,N_k} m_{k,N_k-1} + 2m_{k,N_k} = C_{k,N_k} .$$

The following theorem and its proof are on the same pattern as Theorem II.1.1.

Theorem II.2.1

Let $f(x) \in C[a,b]$. Let $S_{\Delta_k}(x) \in C[a,b]$ be the spline

function interpolating to $f(x)$ at the joint (2.1.1). Suppose $||\Delta_k|| \to 0$ as $k \to \infty$, $R_{\Delta_k} \leq \beta < \infty$ and S satisfies the end conditions (2.2.1) (iii) and (2.2.2)(iii) with $\sup_k \max [|\lambda_{k,o}|, |U_{k,N_k}|] < 2$ and $||\Delta_k||^2 (|d_{k,o}| + |d_{k,N_k}|) \to 0$ as $k \to \infty$. Then for all $x \in [a,b]$ we have

$$|f(x) - S_{\Delta_k}(x)| \leq K\omega(f;||\Delta_k||) \quad .$$

where $K = \frac{3}{2}[\frac{1}{3^{5/2}} \beta^2 \max ((2-\lambda_{k,o})^{-1}, (2-U_{k,N_k})^{-1}, 1) + 1]$

We turn now to the case of general end conditions.

Theorem II.2.2

Let $f(x) \in C'[a,b]$. Let $S_{\Delta_k}(x) \in C'[a,b]$ be the spline interpolating to $f(x)$ at the joints (2.1.1) with $||\Delta_k|| \to 0$ as $k \to \infty$, satisfying end conditions (2.2.3)(iii) and (2.2.4)(iii) with $\inf_k (4-U_{k,o}) > 0$, $\inf_k (4-\lambda_{k,N_k}) > 0$, and $U_{k,o}$ and λ_{k,N_k} bounded as $k \to \infty$.

(a) If $\varepsilon_k' = C_{k,o} - (2 + U_{k,o})f'(a) \to 0$ and

$$\varepsilon_k'' = C_{k,N_k} - (2 + \lambda_{k,N_k})f'(b) \to 0 \quad \text{as} \quad k \to \infty ,$$

then $S'_{\Delta_k}(x)$ converges uniformly to $f'(x)$ on $[a,b]$ and

$$[S^{(r)}_{\Delta_k}(x) - f^{(r)}(x)] = o(||\Delta_k||^{1-r}), \qquad (r = 0,1) \tag{2.2.5}$$

uniformly with respect to x in $[a,b]$.

(b) If $C_{k,o}$ and C_{k,N_k} are bounded as $k \to \infty$, then (2.2.5) is valid on any closed subinterval of $a < x < b$ and $\{S_{\Delta_k}(x)\} \to f(x)$ uniformly on $[a,b]$.

Theorem II.2.3

Let $f(x) \in C^2[a,b]$. Let $S_{\Delta_k}(x)$ be a spline interpolating to $f(x)$ at the joints (2.1.1) with $||\Delta_k|| \to 0$ as $k \to \infty$. Suppose $S_{\Delta_k}(x)$ satisfies end conditions (2.2.1)(iii) and (2.2.2)(iii) with $\inf_k (4-\lambda_{k,o}) > 0$, $\inf_k (4-U_{k,N_k}) > 0$, and $\lambda_{k,o}$ and U_{k,N_k} bounded as $k \to \infty$.

(a) If $\varepsilon_k' = d_{k,o} - (2 + \lambda_{k,o})f''(a) \to 0$ and

$$\varepsilon_k' = d_{k,N_k} - (2 + U_{k,N_k})f''(b) \to 0 \quad \text{as} \quad k \to \infty$$

then $S''_{\Delta_k}(x)$ converges uniformly to $f(x)$ on $[a,b]$ and we have

$$[S^{(r)}_{\Delta_k}(x)-f^{(r)}(x)] = o(||\Delta_k||^{2-r}) , \quad (r = 0,1,2) \qquad (2.2.6)$$

uniformly with respect to x in $[a,b]$.

(b) If $d_{k,o}$ and d_{k,N_k} are bounded as $k \to \infty$ then (2.2.6) is valid on any closed subinterval of $a < x < b$. Moreover, $\{S'_{\Delta_k}(x)\}$ and $\{S_{\Delta_k}(x)\}$ converges uniformly to $f'(x)$ and $f(x)$ respectively on $[a,b]$.

§II.3 Convergence on the Real Line.

Let $\Delta : \ldots < x_{-1} < x_o < x_1 < \ldots$ be a mesh on $(-\infty,\infty)$. A cubic spline $S_\Delta(f;x)$ interpolating to $f(x)$ at the joints x_j of the meash Δ is given in (x_{i-1},x_i) by

$$S_\Delta(f;x) = [M_{i-1}(x_i-x)^3 + M_i(x-x_{i-1})^3]\frac{1}{6h_i}$$

$$+ [\frac{f(x_i)}{h_i} - M_i \frac{h_i}{6}](x-x_{i-1})+[\frac{f(x_{i-1})}{h_i} - M_{i-1}\frac{h_i}{6}](x_i-x)$$

(2.3.1)

where $M_j = S''_\Delta(f;x_j)$.

Since $S'_\Delta(f;x) \in C(-\infty,\infty)$, the following infinite system of equations must be satisifed:

$$\frac{1}{6}h_iM_{i-1} + \frac{1}{3}(h_i+h_{i+1})M_i + \frac{1}{6}h_{i+1}M_{i+1} = \frac{f(x_{i+1})-f(x_i)}{h_{i+1}}$$

$$-\frac{f(x_i)-f(x_{i-1})}{h_i} \equiv d_i\ , \qquad (i = 0, \underline{+}1, \underline{+}2, \ldots)\ . \tag{2.3.2}$$

These equations can be written in matrix form:

$$AM = d\ , \tag{2.3.3}$$

where A is the coefficient matrix of the equations (2.3.2)

$$M = (\ldots, M_{-1}, M_o, M_1, \ldots)^T\ ,$$

$$d = (\ldots, d_{-1}, d_o, d_1, \ldots)^T\ .$$

If we require $\delta_\Delta > 0$, (here $\delta_\Delta = \inf_i h_i$) then $A = (a_{ij})$ is a symmetric matrix, which is strongly diagonally dominant in the sense that

$$\inf_i \{|a_{ii}| - \sum_{j\neq 1} |a_{ij}|\} \geq \frac{1}{3}\delta_\Delta > 0\ . \tag{2.3.4}$$

Moreover, $S_\Delta(f;x)$ exists and is unique if and only if equation (2.3.3)

has a unique solution. In addition, the convergence results of §II.1 and §II.2 can be obtained for the interval $(-\infty,\infty)$ if

$$||A^{-1}||_{\infty} = \sup_j \sum_j |b_{ij}| < K/\delta_{\Delta} \quad , \tag{2.3.5}$$

where K is a constant independent of Δ and

$$A^{-1} = (b_{ij}) \quad . \tag{2.3.6}$$

The following theorem, due to Ahlberg and Nilson [2], establishes the existence of $S_{\Delta}(f;x)$ if $||\Delta|| < \infty$ and $\delta_{\Delta} > 0$. It shows that (2.3.5) is satisfied, hence the convergence results mentioned above can be derived.

Theorem II.3.1

Let $A = (a_{ij})$, $(i,j = 0,1,\ldots)$ be a real symmetric matrix for which

$$\inf_i \{|a_{ii}| - \sum_{j\neq i} |a_{ij}|\} \equiv \delta > 0 \quad .$$

$$\sup_i \sum_{j=o}^{\infty} |a_{ij}| = ||A||_{\infty} < \infty \quad .$$

Under these conditions A^{-1} exists and, with the notation $A^{-1} = (b_{ij})$, we have

$$||A^{-1}||_{\infty} = \sup_i \sum_{j=o}^{\infty} |b_{ij}| < \frac{1}{\delta} \quad . \tag{2.3.7}$$

§II.4 Local Convergence

In this section we shall examine the case in which $f(x)$ is defined and continuous on a closed interval $[a,b]$ but $f''(x)$ is not assumed to be continuous or even to exist in the interval. For convenience we will restrict $f(x)$ to be a periodic function over the closed interval. Let the subdivision of the interval be the same as (2.1.1). The next two theorems may be considered as corollaries of Theorem II.2.2 and Theorem II.2.3 respectively.

Theorem II.4.1

Suppose $f'(x)$ exists and is bounded in $[a,b]$. Suppose $f'(x)$ is continuous on $[a',b']$ with $a < a' < b' < b$. Then $\{S'_{\Delta_k}(x)\} \to f'(x)$ uniformly on $[a',b']$ and $\{S_{\Delta_k}(x)\} \to f(x)$ uniformly in $[a,b]$ as $k \to \infty$.

Theorem II.4.2

Suppose $f''(x)$ exists and is bounded in $[a,b]$. Suppose $f''(x)$ is continuous on $[a',b']$ with $a < a' < b' < b$. Then $\{S''_{\Delta_k}(x)\} \to f''(x)$ uniformly on $[a',b']$ and $\{S^{(r)}_{\Delta_k}(x)\} \to f^{(r)}(x)$ uniformly on $[a,b]$ as $k \to \infty$ for $r = 0,1$.

Define

$$\delta^2_{k,i} = \left[\frac{f(x_{k,i+1})-f(x_{k,i})}{h_{k,i+1}} - \frac{f(x_{k,i})-f(x_{k,i-1})}{h_{k,i}}\right]/\tfrac{1}{2}(h_{i+1}+h_i)$$

$$H_k(I) = \max_i[h_{k,i} \mid x_{k,i-1} \in I \quad \text{or} \quad x_{k,i} \in I] \quad ,$$

$$\Lambda_k(I) = \max_i\left[\left|\tfrac{1}{2} - \lambda_{k,i}\right| \,\Big|\, x_{k,i} \in I \quad \text{and} \quad \lambda_{k,i} = \frac{h_{k,i+1}}{h_{k,i}+h_{k,i+1}}\right] .$$

We then have

the following two theorems which are due to Ahlberg and Nilson [1].

Theorem II.4.3

If $f''(x)$ is continuous in an interval $I : (x^* - \frac{h}{2}, x^* + \frac{h}{2})$ about x^*; if $\delta^2_{k,i}$ (the second difference quotients) are uniformly bounded and if $\Lambda_k(I) \to 0$, $H_k(I) \to 0$ as $k \to \infty$, then $S''_{\Delta_k}(x)$ converges uniformly to $f''(x)$ on any closed interval contained in I.

The following theorem assumes only that $f''(x)$ exists at a single point:

Theorem II.4.4

Let $f''(x)$ exist at $x^* \varepsilon [a,b]$ and let the second difference quotients $\delta^2_{k,j}$ be uniformly bounded on the interval $[a,b]$. Let the sequence of meshes have the property: Given x^*, n, $\varepsilon > 0$, and $\delta > 0$ there exists N sufficiently large, such that for $k > N$ there are at least n mesh points of the kth mesh in each of the intervals $(x^* - \delta, x^*)$, $(x^*, x^* + \delta)$ and that $\left|\frac{h_i}{h_{i+1}} - 1\right| < \varepsilon$ for this set of $2n$ (or $2n+1$) mesh points. Then $S''_{\Delta_k}(x^*)$ (as determined by the spline) converges to $f''(x^*)$ as $k \to \infty$.

§II.5 Convergence of Interpolatory Splines

In the previous section all the interpolatory conditions were imposed at the mesh points. In the present section, we quote convergence properties of periodic cubic splines which interpolate to a given function at one or more inner points of the given mesh intervals. The theorems in this section are the results of A. Meir

and A. Sharma [19]. In Theorem II.5.1 error bounds are obtained for cubic splines which interpolate at one point in each mesh interval, in Theorem II.5.2 for splines which interpolate at two points in each mesh interval. For the sake of simplicity, we consider only equidistant joints in the first theorem. In this second theorem, this restriction is not needed.

Let

$$\Delta : 0 = x_o < x_1 < \ldots < x_n = 1 \tag{2.5.1}$$

be any subdivision of $[0,1]$. For a given λ , $0 \leq \lambda \leq 1$ denote $t_i = x_i + \lambda h_i$, $0 \leq i \leq n-1$. For any given n-tuple $(\alpha_o,\ldots,\alpha_{n-1})$ of real numbers there exists a unique 1-periodic cubic spline $S(x) \varepsilon C^2[0,1]$ with joints Δ such that $S(t_i) = \alpha_i$, $0 \leq i \leq n-1$. The existence and uniqueness of such a spline can be proved by the method of Ahlberg, Nilson, and Walsh [5].

If the α_i's are the values of a 1-periodic function $f(x)$, the following convergence theorem holds for equidistant joints:

Theorem II.5.1

Let $f(x) \varepsilon C^2[0,1]$ be a 1-perodic function. Let $S(x) \varepsilon C^2[0,1]$ be the 1-periodic cubic spline with joints $\frac{i}{n}$, satisfying

$$S\left(\frac{i+\lambda-1}{n}\right) = f\left(\frac{i+\lambda-1}{n}\right) , \qquad i = 1,2,\ldots,n ,$$

where $0 \leq \lambda \leq \frac{1}{3}$ or $\frac{2}{3} \leq \lambda \leq 1$.

Then we have, for all $x \varepsilon [0,1]$,

$$|S^{(r)}(x) - f^{(r)}(x)| \leq 15\, n^{r-2}\, \omega(f''; \tfrac{1}{n}) , \qquad (r = 0,1,2) .$$

An analogous result where interpolation takes place between the joints rather than at the joints is due to Subotin [27].

In the next theorem the spline interpolates a function at two points in each subinterval. Let (2.5.1) be a given subdivision of $[0,1]$; ℓ, m $(0 \leq m < \ell \leq 1)$ given real numbers with $\frac{1}{2} < \ell+m < \frac{3}{2}$ and $(\alpha_1,\ldots,\alpha_n)$ and $(\beta_1,\ldots,\beta_n)$ are given n-tuples of reals. Then there exists a unique 1-periodic cubic spline $S(x) \in C^1[0,1]$ with joints (2.5.1) such that

$$S(\xi_i) = \alpha_i, \quad S(\eta_i) = \beta_i , \qquad i = 1,2,\ldots,n ,$$

where $\xi_i = mx_i + (1-m)x_{i-1}$, $\eta_i = \ell x_i + (1-\ell)x_{i-1}$.

The proof of existence and uniqueness can be carried out along the usual lines [28]. The main result here is:

Theorem II.5.2

Let $f(x) \in C^1[0,1]$ be a 1-periodic function. Let $S(x) \in C^1[0,1]$ be the 1-periodic cubic splines with joints (2.5.1), satisfying

$$S(\xi_i) = f(\xi_i) = \alpha_i ; \quad S(\eta_i) = f(\eta_i) = \beta_i , \qquad i = 1,2,\ldots,n .$$

Then we have for all $x \in [0,1]$

$$|S^{(r)}(x)-f^{(r)}(x)| \leq K(\ell,m)\omega(f'';||\Delta||)||\Delta||^{1-r} , \qquad (r = 0,1) ,$$

where $K(\ell,m)$ depends on ℓ and m only.

§II.6 Uniform Approximation by Cubic Splines

Let C denote the Banach space (with supremum norm) of all continuous, real valued functions on $[0,1]$ with the condition $f(0) = f(1)$. For $n = 1,2,\ldots,$ let

$$\Delta_n : 0 = x_{n,o} < x_{n,1} < \ldots < x_{n,N_n} = 1 \tag{2.6.1}$$

be subdivisions of $[0,1]$. Define S_n to be the subspace of C whose numbers are the periodic cubic spline functions with joints $x_{n,i}$. To each $f \varepsilon C$ there corresponds a uniquely determined element $S \varepsilon S_n$ with the interpolating property $S(x_{n,i}) = f(x_{n,i})$, $i = 0,1,\ldots,N_n$. The mapping $L_n : f \to S$ is a linear idempotent operator from C onto S_n. Cheney and Schurer [9] obtained estimates for the operator norms

$$||L_n|| = \sup\{||L_nf|| : ||f|| = 1, f \varepsilon C\}$$

in terms of the spacing numbers $h_{n,i} = x_{n,i} - x_{n,i-1}$. In this connection, the following question is of interest: What are the conditions on the sequence $\{\Delta_n\}$ of partitions which imply that $\lim_{n \to \infty} ||L_nf-f|| = 0$ for $f \varepsilon C$?

Define

$$K_n = \max \{\frac{h_{n,i}}{h_{n,j}} : i,j = 1,2,\ldots,N_n\} \quad ,$$

and

$$P_n = \max \{\frac{h_{n,i}}{h_{n,j}} : |i-j| = 1 \quad \text{and} \quad i,j = 1,2,\ldots,N_n\} .$$

Sharma and Meir [26] have shown that

$$K_n \leq K < \infty \tag{2.6.2}$$

is a sufficient condition for

$$\lim_n ||L_n f - f|| = 0 \quad \text{for} \quad f \in C \ . \tag{2.6.3}$$

The next theorem of Cheney and Schurer [9] shows that (2.6.3) may hold even if (2.6.2) is false.

Theorem II.6.1

Let $n = 2K+1$ and $\frac{1}{2} < \theta < 1$. Determine h by the equation $h + 2\theta h + 2\theta^2 h + \ldots + 2\theta^k h = 1$ and let the division of the interval $[0,1]$ be defined by (2.5.1) with $h_i = h_{2k+2-i} = \theta^{k+1-i} h$ for $i = 1,2,\ldots,k+1$. Then for the operator L_n, we have $1 \leq ||L_n|| \leq 19(2\theta-1)$. It is easy to see that in this case $k_n \to \infty$ as $n \to \infty$ while $\sup_n ||L_n|| < \infty$.

Remark II.6.1

It was proved by Cheney and Schurer [10] that if $\lim\limits_{n\to\infty} ||\Delta_n|| = 0$ then the following conditions are equivalent:

(1) $L_n f \to f$ (uniformly for all $f \in C$),

(2) $\lim\limits_{n\to\infty} \sup_n ||L_n|| < \infty$.

In [20] Meir and Sharma have shown that if $P_n \leq P < \sqrt{2}$ then (2.6.3) holds. Cheney and Schurer [9] proved the same under the weaker condition $P_n \leq P < 2$. C.A. Hall [16] extends this results to the case when $P_n \leq P < 1 + \sqrt{2}$.

In the next theorem, due to M. Marsden [18] it is shown that the inequality $P_n \leq P < 2.439^+$ is a sufficient condition for

(2.6.3) to hold:

Theorem II.6.2

If $P_n \leq P < 2.439^+$ for all n. Then

$$||L_n|| \leq \frac{2(1+P)(2+P)(1+P+P^2)}{6P+7P^2+P^3-2P^4}, \qquad n = 1,2,\ldots \quad .$$

On the other hand, Marsden's next theorem shows that $P_n \leq P < \infty$ is not a sufficient condition in general for (2.6.3) to hold.

Theorem II.6.3

For each fixed $P > (3 + \sqrt{5})/2$ there exists a sequence of meshes $\{\Delta_n\}$ for which $P_n \leq P$ for all n, while $\lim_n \sup ||L_n|| = \infty$.

Remark II.6.2

The question whether the inequality $P_n \leq P$ is a sufficient condition for (2.6.3) if P is between 2.439^+ and 2.62^- is still open.

CHAPTER THREE

QUINTIC SPLINES

It is natural to investigate possible extensions of the convergence properties of cubic splines to quintic splines. A complete array of convergence properties analogous to those presented for cubic splines in Chapter II is not yet available for quintic splines in general. If $f(x)$ is of class $C^q[a,b]$ $(0 \le q \le 5)$ and $S_\Delta(x)$ is the quintic spline of interpolation to $f(x)$ on Δ_k, satisfying certain end conditions, and for the sequence of meshes $\{\Delta_k\}$ we have $\lim_{k \to \infty} ||\Delta_k|| = 0$, we might expect that under certain conditions

$$f^{(p)}(x) - S_{\Delta_k}^{(p)}(x) = o(||\Delta_k||^{q-p}) \quad ,$$
$$(0 \le p \le q \le 5) \quad .$$

§III.1 Fundamental Convergence Theorem

In the first part of this section we quote a sequence of results due to Ahlberg, Nilson and Walsh, which provide the basic convergence results for periodic splines on uniform meshes.

Theorem III.1.1

Let $f(x)$ be of class $C^q(-\infty,\infty)$ $(0 \le q \le 4)$ and of period 1. Let $\{\Delta_k\}$ be a sequence of uniform meshes on $[0,1]$ with $||\Delta_k|| \to 0$ as $k \to \infty$. If $S_{\Delta_k}(x)$ is the periodic quintic spline interpolating to $f(x)$ on Δ_k, then we have uniformly on $[0,1]$

$$[f^{(p)}(x)-S^{(p)}_{\Delta_k}(x)] = o(||\Delta_k||^{q-p}) \;, \qquad 0 \le p \le q \;.$$

Theorem III.1.2

Let $f(x) \in C^5(-\infty,\infty)$ with period 1. Let $\{\Delta_k\}$ be a sequence of uniform meshes on $[0,1]$ with $||\Delta_k|| \to 0$ as $k \to \infty$. Let $S_{\Delta_k}(x)$ be the periodic quintic spline interpolating to $f(x)$ at the mesh points. Then we have uniformly on $[0,1]$

$$[f^{(p)}(x)-S^{(p)}_{\Delta_k}(x)] = o(||\Delta_k||^{5-p}) \;, \qquad p = 0,1,\ldots,5 \;.$$

Theorem III.1.3

Suppose $f(x) \in C^6[0,1]$ and has period 1. Let $\{\Delta_k\}$ be a sequence of uniform meshes on $[0,1]$ with $||\Delta_k|| \to 0$ as $k \to \infty$. If $S_{\Delta_k}(x)$ is the periodic quintic spline of interpolation to $f(x)$ on Δ_k, then

$$\lim_{k\to\infty} [\max \left| \frac{S^{(5)}_{\Delta_k}(x_{k,j}+)-S^{(5)}_{\Delta_k}(x_{k,j}-)}{h_{k,j}} - \frac{f^{(4)}(x_{k,j})}{2} \right|] = 0 \;.$$

The following theorem of F. Schurer (see [22], [23]) provides more precise results for the error bounds:

Theorem III.1.4

Let

$$\Delta : 0 = x_o < x_1 < \ldots < x_n = 1 \qquad (3.1.1)$$

be a mesh on $[0,1]$ with $x_i - x_{i-1} = h_i = \frac{1}{n}$, $i = 1,2,\ldots,n$. Let $f(x) \in C(-\infty,\infty)$ and have period 1. Let $S(x) \in C^4(-\infty,\infty)$ be the periodic qunitic spline interpolating to $f(x)$ at the joints (3.1.1), Then for all x

$$|S(x) - f(x)| \leq 2\,\frac{7}{48}\,\omega(f; \tfrac{1}{n}) \quad .$$

Theorem III.1.5

Let $f(x) \in C^1(-\infty,\infty)$ and have period 1. Let $S(x) \in C^4(-\infty,\infty)$ be the periodic quintic spline interpolating to $f(x)$ at the joints (3.1.1). Then for all x

$$|S^{(r)}(x) - f^{(r)}(x)| \leq 73\,\frac{7}{24}\; n^{r-1}\,\omega(f'; \tfrac{1}{n}) \;, \qquad (r = 0,1).$$

<u>Proof</u>: For the proof of Theorem III.1.5 we need the following lemma (see p. 497 [22]):

Lemma III.1.1

Let $f(x)$ be a continuous function with period 1. Let $S(x) \in C^4(-\infty,\infty)$ be the periodic quintic spline interpolating to $f(x)$ at the joints of (3.1.1). Setting $\lambda_i = S_i'$, $U_i = S_i''$, $m_i = S_i^{(3)}$ and $M_i = S_i^{(4)}$. We have

$$\max_i |\lambda_i| \leq \frac{25}{6}\,\omega(f'; \tfrac{1}{n}) \;, \qquad \max_i |U_i| \leq \frac{145}{12}\; n\,\omega(f'; \tfrac{1}{n}) \;,$$

$$\max_i |m_i| \leq 20\, n^2\,\omega(f'; \tfrac{1}{n}), \qquad \max_i |M_i| \leq 160\, n^3\,\omega(f'; \tfrac{1}{n}) \;.$$

The proof of this lemma is similar to that given in [22].

Proof of Theorem III.1.5: From (p. 10, [5]), we have for

$\frac{i-1}{n} \leq x \leq \frac{i}{n}$,

$$S''(x) = \frac{n}{6}[M_{i-1}(x_i-x)^3 + M_i(x-x_{i-1})^3]$$
$$+ n[S''_i(x-x_{i-1}) + S''_{i-1}(x_i-x)]$$
$$- \frac{1}{6n}[M_i(x-x_{i-1}) + M_{i-1}(x_i-x)] \quad . \tag{3.1.2}$$

If we integrate (3.1.2) and evaluate the constant of integration, we obtain the equation

$$S'(x) = \frac{n}{24}[M_i(x-x_{i-1})^4 - M_{i-1}(x_i-x)^4]$$
$$+ \frac{n}{2}[U_i(x-x_{i-1})^2 - U_{i-1}(x_i-x)^2]$$
$$- \frac{1}{12n}[M_i(x-x_{i-1})^2 - M_{i-1}(x_i-x)^2] \quad . \tag{3.1.3}$$

Using (3.1.3) and Lemma III.1.1, we have

$$|S'(x) - f'(x)| \leq |S'(x_i) - f'(x_i)| + |f'(x_i) - f'(x)|$$
$$+ \frac{7}{24n^3} \max_i |M_i| + \frac{3}{2n} \max_i |U_i|$$
$$\leq |S'(x_i) - f'(x_i)| + \omega(f'; \tfrac{1}{n})$$
$$+ \frac{7}{24} \cdot 160\, \omega(f'; \tfrac{1}{n}) + \frac{3}{2} \cdot \frac{125}{12}\, \omega(f'; \tfrac{1}{n}). \tag{3.1.4}$$

This implies (see p. 496, [22]), that

$$\lambda_{i-2} + 26\,\lambda_{i-1} + 66\,\lambda_i + 26\,\lambda_{i+1} + \lambda_{i+2}$$

$$= 5[f'(\xi_i) + 11\,f'(\eta_i) + 11\,f'(\zeta_i) + f'(\theta_i)] \qquad (3.1.5)$$

$$\frac{i+1}{n} < \xi_i < \frac{i+2}{n}\ , \qquad \frac{i}{n} < \eta_i < \frac{i+1}{n}\ ,$$

$$\frac{i-1}{n} < \zeta_i < \frac{i}{n}\ , \qquad \frac{i-2}{n} < \theta_i < \frac{i-1}{n}\ .$$

If we set $a_i = S'_i - f'_i$, then

$$a_{i-2} + 26\,a_{i-1} + 66\,a_i + 26\,a_{i+1} + a_{i+2}$$

$$= 5f'(\xi_i) + 55\,f'(\eta_i) + 55f'(\xi_i) + 5f'(\theta_i) - f'_{i-2}$$

$$- 26\,f'_{i-1} - 66\,f'_i - 26f'_{i+1} - f'_{i+2}$$

$$= (f'(\xi_i)-f'_{i-2}) + 4(f'(\xi_i)-f'_{i-1}) + 22(f'(\eta_i)-f'_{i-1})$$

$$+ 33(f'(\eta_i)-f'_i) + 33(f'(\zeta_i)-f'_i) + 22(f'(\zeta_i)$$

$$- f'_{i+1}) + 4(f'(\theta_i)-f'_{i+1}) + f'(\theta_i)-f_{i+2}\ . \qquad (3.1.6)$$

Assume now $\max_i |a_i| = a_k$. Then

$$66\,|a_k| \leq 120\,\omega(f';\tfrac{1}{n}) + 54\,|a_k|\ .$$

Hence

$$|a_k| \leq 10\,\omega(f';\,\tfrac{1}{n})\ . \qquad (3.1.7)$$

Applying (3.1.7) to (3.1.4), we obtain for all x

$$|S'(x) - f'(x)| \leq 73\,\frac{7}{24}\ \omega(f';\,\tfrac{1}{n})\ .$$

Suppose now $\frac{i-1}{n} \leq x \leq \frac{i}{n}$. Since $f_{i-1} = S_{i-1}$ and $f_i = S_i$, for a suitable σ_i in $(\frac{i-1}{n}, \frac{i}{n})$ we have, $f'(\sigma_i) - S'(\sigma_i) = 0$.

Hence

$$|f(x) - S(x)| = \left| \int_{\sigma_i}^{x} (f'(t) - S'(t))dt \right|$$

$$\leq 73 \frac{7}{24} (\frac{1}{n}) \, \omega(f'; \frac{1}{n}) \quad ,$$

uniformly in $[0,1]$ and the result is proved.

Theorem III.1.6

Suppose $f \in C^2(-\infty,\infty)$ and has period 1. Let $S(x) \in C^4(-\infty,\infty)$ be the periodic quintic spline interpolating to $f(x)$ at the joints of (3.1.1). Then for all x we have

$$|S^{(r)}(x) - f^{(r)}(x)| \leq (\frac{75}{3^{5/2}} + \frac{3}{2}) \, (\frac{1}{n})^{2-r} \, \omega(f''; \frac{1}{n}) \quad ,$$

$$r = 0,\ldots,2 \, .$$

Proof: For the sake of brevity we set $S_i^{(r)} = S^{(r)}(x_i)$, $f_i^{(r)} = f^{(r)}(x_i)$, $(r = 0,1,2,3)$, and $M_i = S^{(4)}(x_i)$, $i = 1,2,\ldots,n$. By the linearity of the 4th derivative of $S(x)$ we have on $[\frac{i-1}{n}, \frac{i}{n}]$

$$S^{(4)}(x) = n[M_{i-1}(x_i-x) + M_i(x-x_{i-1})] \quad . \tag{3.1.8}$$

Integrating twice and evaluating the constants of integration, we have on account of the continuity of $S^{(3)}(x)$ at $\frac{i}{n}$,

$$\frac{1}{2} M_{i-1} + 2M_i + \frac{1}{2} M_{i+1} = 3 \, n^2[S''_{i+1} - 2S''_i + S''_{i-1}] \quad . \tag{3.1.9}$$

The defining equations in matrix form are

$$\begin{bmatrix} 2 & \frac{1}{2} & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 2 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \cdots & 0 & \frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ M_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{bmatrix} \tag{3.1.10}$$

where the di's represent the right hand side of (3.1.9). If we denote by A the coefficient matrix in (3.1.10), then clearly $||A^{-1}|| < 1$. If $A^{-1}_{i,j}$ are the elements of the inverse of the coefficient matrix A in (3.1.10), then we have

$$M_j = \frac{3}{2} \sum_{i=1}^{n} A^{-1}_{ji} n^2 [S''_{i+1} - 2S''_i + S''_{i-1}] \quad . \tag{3.1.11}$$

Now, if f belongs, respectively, to the classes C^4, C^3, C^2, C^1 and has period 1, while S is the interpolating quintic spline associated with f, then it follows from [22] that

$$\max |S_i^{(r)} - f_i^{(r)}| \le C_r \, \omega(f^{(r)}; \tfrac{1}{n}) \quad , \quad (r = 1,2,3,4) \tag{3.1.12}$$

where $C_1 = 10$, $C_2 = 12$, $C_3 = 23$, and $C_4 = 25$. Using this, we obtain

$$\begin{aligned} |S''_{i+1} - 2S''_i + S''_{i-1}| &\le |S''_{i+1} - f''_{i+1}| + 2|S''_i - f''_i| + |S''_{i-1} - f''_{i-1}| \\ &\quad + |f''_{i+1} - f''_i| + |f''_i - f''_{i-1}| \\ &\le 50 \, \omega(f''; \tfrac{1}{n}) \quad . \end{aligned}$$

Therefore, by (3.1.11)

$$\frac{1}{n^2}(|M_{j-1}|+|M_j|) \leq 75\,\omega(f''; \frac{1}{n}) \quad .$$

From (p. 23, [5]), we also have

$$|S''(x) - f''(x)| \leq \frac{1}{3^{5/2}} \cdot 75\,\omega(f''; \frac{1}{n}) + \frac{3}{2}\,\omega(f''; \frac{1}{n})$$

$$= (\frac{75}{3^{5/2}} + \frac{3}{2})\,\omega(f''; \frac{1}{n}) \quad .$$

Let now $\frac{i-1}{n} \leq x \leq \frac{i}{n}$. Since $f_i = S_i$ and $f_{i-1} = S_{i-1}$ for a suitable η_i in $(\frac{i-1}{n}, \frac{i}{n})$ we have $f'(\eta_i) - S'(\eta_i) = 0$.

Hence

$$|f'(x) - S'(x)| = \int_{\eta_i}^{x} [f''(t) - S''(t)]dt$$

$$\leq (\frac{75}{3^{5/2}} + \frac{3}{2})(\frac{1}{n})\,\omega(f''; \frac{1}{n}) \quad .$$

Thus,

$$|f(x) - S(x)| = |\int_{x_{i-1}}^{x} [f'(t) - S'(t)]dt|$$

$$\leq (\frac{75}{3^{5/2}} + \frac{3}{2})(\frac{1}{n})^2\,\omega(f''; \frac{1}{n}) \quad ,$$

uniformly in $[0,1]$, which completes the proof of this theorem.

Theorem III.1.7

Suppose $f(x) \varepsilon C^3(-\infty,\infty)$ and has period 1. Let $S(x) \varepsilon C^4(-\infty,\infty)$ be the periodic quintic spline interpolating to $f(x)$ at the joints of (3.1.1). Then for all x we have

$$|S^{(r)}(x) - f^{(r)}(x)| \leq 47\frac{1}{2}\left(\frac{1}{n}\right)^{3-r} \omega(f''; \frac{1}{n}) ,$$

$$(r = 0,1,2,3) .$$

Proof: Since $S^{(3)}(x)$ is a piecewise quadratic in $[0,1]$, we have for $\frac{i-1}{n} \leq x \leq \frac{i}{n}$,

$$S^{(3)}(x) = \alpha_i(x - \frac{i-1}{n})(x - \frac{i}{n}) + n[(\frac{i}{n} - x) S_{i-1}^{(3)} + (x - \frac{i-1}{n})S_i^{(3)}] \qquad (3.1.13)$$

where $2\alpha_i = S^{(5)}(x)$.

Given $S \in C^4(-\infty,\infty)$, we have for $x \in [\frac{i-1}{n}, \frac{i}{n}]$

$$S^{(4)}(x) = \alpha_i(2x - \frac{2i-1}{n}) + n(S_i^{(3)} - S_{i-1}^{(3)}) . \qquad (3.1.14)$$

From (3.1.14) we obtain for the one sided limits of the derivative:

$$S^{(4)}(x_i+) = -\frac{\alpha_{i+1}}{n} + n(S_{i+1}^{(3)} - S_i^{(3)})$$

$$S^{(4)}(x_i-) = \frac{\alpha_i}{n} + n(S_i^{(3)} - S_{i-1}^{(3)}) . \qquad (3.1.15)$$

The continuity of $S^{(4)}(x)$ at x_i yields, by means of (3.1.15),

$$\alpha_i + \alpha_{i+1} = n^2[S_{i+1}^{(3)} - 2S_i^{(3)} + S_{i-1}^{(3)}] .$$

Therefore, from (3.1.12), we get

$$\frac{|\alpha_i|}{n^2} \leq 94\ \omega(f^{(3)}; \frac{1}{n}) . \qquad (3.1.16)$$

Let now $\frac{i-1}{n} \leq x \leq \frac{i}{n}$. Setting

$$L_1(x) = [(\frac{i}{n} - x)f_{i-1}^{(3)} + (x - \frac{i-1}{n})f_i^{(3)}]n \quad ,$$

$$L_2(x) = [(\frac{i}{n} - x)S_{i-1}^{(3)} + (x - \frac{i-1}{n})S_i^{(3)}]n \quad ,$$

it is easy to see, from elementary considerations and from (3.1.12), that

$$\begin{aligned}|f^{(3)}(x) - S^{(3)}(x)| &\leq |f^{(3)} - L_1(x)| + |L_1(x) - L_2(x)| \\ &\quad + |L_2(x) - S^{(3)}(x)| \\ &\leq \omega(f^{(3)}; \tfrac{1}{n}) + 23\omega(f^{(3)}; \tfrac{1}{n})| \\ &\quad + |L_2(x) - S^{(3)}(x)| \quad .\end{aligned}$$

By (3.1.13), we have

$$\begin{aligned}|L_2(x) - S^{(3)}(x)| &\leq |\alpha_i| \; |(x - \tfrac{i-1}{n})(x - \tfrac{i}{n})| \\ &\leq \frac{1}{4n^2} |\alpha_i| \quad . \end{aligned} \tag{3.1.17}$$

Combining this inequality with (3.1.16) and (3.1.17), we have

$$\begin{aligned}|f^{(3)}(x) - S^{(3)}(x)| &\leq 24\,\omega(f^{(3)}; \tfrac{1}{n}) + \frac{94}{4}\,\omega(f^{(3)}; \tfrac{1}{n}) \\ &= \frac{95}{2}\,\omega(f^{(3)}; \tfrac{1}{n}) \quad .\end{aligned}$$

This proves Theorem III.1.7 with $r = 3$. The proofs for $r = 0,1,2$ run exactly as in Theorem III.1.6.

The following theorem is due to F. Schurer [22].

Theorem III.1.8

Let $f \in C^4(-\infty,\infty)$ and has period 1. Let $S \in C^4(-\infty,\infty)$ be the periodic quintic spline interpolating to $f(x)$ at the joints

(3.1.1). Then for all x_i,

$$|S^{(4)}(x) - f^{(4)}(x)| \le 26\,\omega(f^{(4)}; \tfrac{1}{n}) ;$$

$$|S^{(3)}(x) - f^{(3)}(x)| \le 26(\tfrac{1}{n})\,\omega(f^{(4)}; \tfrac{1}{n}) + 23\,\omega(f^{(3)}; \tfrac{1}{n}) ;$$

$$|S^{(r)}(x) - f^{(r)}(x)| \le 26(\tfrac{1}{n})^{4-r}\,\omega(f^{(4)}; \tfrac{1}{n}) + 23(\tfrac{1}{n})^{3-r}\,\omega(f^{(3)}; \tfrac{1}{n}) + 12(\tfrac{1}{n})^{2-r}\,\omega(f^{(2)}; \tfrac{1}{n}) \quad (r = 0,1,2) .$$

Now we omit the restriction that the joints be equally spaced. Moreover, except where it is otherwise stated, the function $f(x)$ will not be periodic. The following theorem proved by A. Sharma and A. Meir [26] gives a discussion of deficient quintic spline.

Theorem III.1.9

Let $f(x) \in C^3(-\infty,\infty)$ having period 1. If $S(x) \in C^3(-\infty,\infty)$ is the quintic spline with period 1 and with

$$S_i = f_i , \quad S_i' = f_i' , \quad i = 1,2,\ldots,n ,$$

then for all x we have

$$|S^{(r)}(x) - f^{(r)}(x)| \le 26\,||\Delta||^{3-r}\,\omega(f^{(3)}; ||\Delta||) , \quad r = 0,1,2,3 .$$

The next theorem of C.A. Hall [15] yields error bounds for quintic splines interpolating to an $f(x)$ in $C^6[0,1]$.

Theorem III.1.10

Let $f(x) \in C^6[0,1]$. Let $S_{\Delta_k}(x)$ be the quintic spline

interpolating to $f(x)$ at the joints $\Delta_k: 0 = x_{k,0} < x_{k,1} < \ldots < x_{k,N_k} = 1$ with the condition $||\Delta_k|| \to 0$ as $k \to \infty$ and $R_{\Delta_k} \leq \beta < \infty$.

Then

$$|S_{\Delta_k}^{(r)}(x) - f^{(r)}(x)| \leq \varepsilon_r ||f^6|| \, ||\Delta_k||^{6-r} , \qquad r = 0,1,\ldots,5 ,$$

where

$$||f|| = \max\{|f(x)| \mid 0 \leq x \leq 1\}$$

$$\varepsilon_o = \frac{1}{15360}$$

$$\varepsilon_1 = \frac{\sqrt{5}}{30000} + \frac{\sqrt{3}}{12960}$$

$$\varepsilon_2 = \frac{11}{5760}$$

$$\varepsilon_3 = \frac{1}{60} + (\frac{1}{2} + \beta)$$

$$\varepsilon_4 = \frac{1}{60} + (6 + 5\beta^2)$$

and

$$\varepsilon_5 = \frac{1}{6}(3 + \beta^2) \quad .$$

§III.2 Quintic Splines on the Real Line

We now investigate the convergence properties of quintic splines which are defined on $(-\infty,\infty)$ with uniformly spaced mesh points $x_j = jh$, $(j = 0, \pm 1, \ldots)$, $h > 0$. The existence of such spline functions was established by Schoenberg [21]. The following result related to uniform convergence was obtained by Ahlberg and Nilson [3].

Theorem III.2.1

Let $f(x)$ have a bounded and continuous qth derivative

on $(-\infty,\infty)$, $(q = 4,5)$. Let $S_h(x)$ be the quintic spline of interpolation to $f(x)$ at the joints $x_j = jh$, $(j = 0, \pm 1, \ldots)$. Then

$$S_h^{(q)}(x) \to f^{(q)}(x)$$

uniformly, as $h \to 0$, and

$$S_h^{(r)}(x)-f^{(r)}(x) = O(h^{q-r}\omega(f^{(q)};(\tfrac{7}{2} + q)h) \;, \qquad r = 0,1,\ldots,q \;.$$

Local behavior of spline approximations. Let

$$\omega_x(f;\delta) = \sup\{|f(x) - f(y)| \mid \text{all } y,\; |y-x| < \delta\} \;,$$

so that $\omega_x(f;\delta)$ is a local modulus of continuity for $f(x)$ depending only on the behavior of $f(x)$ in a δ-neighborhood of x. The next theorem, due to Ahlberg and Nilson, gives an important result about local convergence.

Theorem III.2.2

Let $f(x)$ have a bounded qth derivative on $(-\infty,\infty)$, $(q = 4,5)$ and let $\omega_x(f^{(q)};\delta) \to 0$ as $\delta \to 0$. Then

$$S_h^{(q)}(x) \to f^{(q)}(x)$$

as $h \to 0$, and

$$|S_h^{(r)}(x) - f^{(r)}(x)| \leq h^{q-r} \; |S_h^{(q)}(x) - f^{(q)}(x)| \quad ,$$

$$\text{for} \qquad r = 0,1,\ldots,q \quad ,$$

where $S_h(x)$ is the quintic spline of interpolation to $f(x)$ at the nodes $x_j = jh$, $(j = 0, \pm 1, \ldots)$.

§III.3 Convergence of Interpolatory Splines

In §II.6 we discussed some theorems about cubic splines where interpolation takes place between the joints rather than at the joints. The object of the present section is to obtain analogous results for quintic splines. Here we obtain error bounds for quintic splines which interpolate at one point in each mesh interval (Theorem III.3.1) and at two points in each mesh interval (Theroem III.3.2). For convenience, we shall assume in the first theorem that the joints are equally spaced. In the second theorem we remove this restriction.

Theorem III.3.1

Let $f(x) \varepsilon C^2(-\infty,\infty)$ having period 1. Let $S(x) \varepsilon C^2(-\infty,\infty)$ be the quintic spline on the joints (3.1.1) with period 1 and with

$$S(x_i) = f(x_i) \quad ;$$

$$S'(\xi_i) = f'(\xi_i) \quad ;$$

$$S''(\xi_i) = f''(\xi_i) \quad , \qquad i = 1,2,\ldots,n \; ,$$

where $\xi_i = \dfrac{i+\lambda-1}{n}$, $0 < \lambda < 1$, λ fixed. Then for all x we have

$$|f^{(r)}(x) - S^{(r)}(x)| < K(\lambda)(\tfrac{1}{n})^{2-r}\,\omega(f''; \tfrac{1}{n}) \; , \qquad r = 0,1,2 \; ,$$

where $K(\lambda)$ is an absolute constant dependent only on λ .

For the proof of this theorem we shall need the following

Lemma

If $P(x)$ is any quintic polynomial in $[a,b]$, then for $0 < \lambda < 1$ the following identity is valid:

$$A(\lambda)P(b) + B(\lambda)P(a) + C(\lambda)hP'(a) + D(x)hP'(a+\lambda h)$$
$$+ E(\lambda)h^2P''(b) + F(\lambda)h^2P''(a)$$
$$+ G(\lambda)h^2P''(a+\lambda h) = 0 \quad , \qquad (3.3.1)$$

where $h = b-a$, $a < x < b$.

$$A(\lambda) = 60\lambda^3(\lambda-1)(2-\lambda)$$

$$B(\lambda) = -60\lambda^3(\lambda-1)(2-\lambda)$$

$$C(\lambda) = 12(5\lambda^4 - 10\lambda^3 + 5\lambda-2)(\lambda-1)$$

$$D(\lambda) = -12(\lambda-1)(5\lambda-2)$$

$$E(\lambda) = \lambda^3(10\lambda^2 - 15\lambda + 6)$$

$$F(\lambda) = -\lambda(\lambda-1)(-20\lambda^3 + 55\lambda^2 - 44\lambda + 12)$$

$$G(\lambda) = \lambda(3\lambda-2)(5\lambda-6) \ .$$

This lemma may be verified directly or by use of Euler-Marclaurin's expansion.

<u>Proof of Theorem III.3.1</u>: For the sake of brevity, set $M_i = S''(\frac{i}{n})$, $S_i' = S'(\frac{i}{n})$, $S_i = S(\frac{i}{n})$, $\alpha_i = S'(\xi_i)$ and $\alpha_i' = S''(\xi_i)$. Then using first with $a = \frac{i}{n}$, $b = \frac{i+1}{n}$, $a + \lambda h = \xi_{i+1}$, next with

$a = \frac{i}{n}$, $b = \frac{i-1}{n}$, $a + \lambda' h = \xi_i$, $\lambda' = 1-\lambda$ and eliminating S_i' from the two equations so obtained, we have, for $i = 1,2,\ldots,n$,

$$\begin{aligned}\bar{A}\, f_{i+1} + \bar{\bar{A}}\, f_{i-1} + (\bar{B} + \bar{\bar{B}}) f_i + \frac{1}{n}\bar{D}\,\alpha_{i+1} - \frac{1}{n}\bar{\bar{D}}\,\alpha_i \\ + \frac{1}{n^2}\bar{E}\, M_{i+1} + \frac{1}{n^2}\bar{\bar{E}}\, M_{i-1} + \frac{1}{n^2}(\bar{F} + \bar{\bar{F}}) M_i \\ + \frac{1}{n^2}\bar{G}\,\alpha'_{i+1} + \frac{1}{n^2}\bar{\bar{G}}\,\alpha'_i = 0 \quad ,\end{aligned} \tag{3.3.2}$$

where we set

$$\bar{A} = A(\lambda)\cdot\lambda \quad , \qquad \bar{B} = B(\lambda)\cdot\lambda\ , \quad \ldots \quad \text{etc.}\ ,$$

$$\bar{\bar{A}} = A(1-\lambda)\cdot(1-\lambda) \quad , \qquad \bar{\bar{B}} = B(1-\lambda)\cdot(1-\lambda)\ , \quad \ldots \quad \text{etc.}\ ,$$

clearly $\bar{C} = \bar{\bar{C}} = 12\lambda(1-\lambda)(5\lambda^4 - 10\lambda^3 + 5\lambda - 2)$.

A further simplication yields the following system of three term relations:

$$\begin{aligned}\frac{\bar{E}}{n^2} M_{i+1} + \frac{F + \bar{\bar{F}}}{n^2} M_i + \frac{\bar{\bar{E}}}{n^2} M_{i-1} \\ = -\ [\bar{A}(f_{i+1} - f_i) + \frac{1}{n}\bar{D}\,\alpha_{i+1} + \frac{1}{n^2}\bar{G}\,\alpha_{i+1}] \\ -\ [\bar{\bar{A}}(f_{i+1} - f_i) - \frac{1}{n}\bar{\bar{D}}\,\alpha_i + \frac{1}{n^2}\bar{\bar{G}}\,\alpha'_i] \quad .\end{aligned} \tag{3.3.3}$$

Using the fact that $\bar{A} + \bar{C} + \bar{D} = 0$ and $\bar{\bar{A}} + \bar{\bar{C}} + \bar{\bar{D}} = 0$ we obtain

$$\frac{\bar{E}}{n^2} M_{i+1} + \frac{\bar{F} + \bar{\bar{F}}}{n^2} M_i + \frac{\bar{\bar{E}}}{n^2} M_{i-1}$$

$$= - [\bar{A}(f_{i+1} - f_i + \frac{1}{n} f'_i) + \frac{1}{n} \bar{D}(\alpha_{i+1} - f') + \frac{1}{n^2} \bar{G}\alpha'_{i+1}]$$

$$- [\bar{\bar{A}}(f_{i+1} - f_i + \frac{1}{n} f'_i) - \frac{1}{n} \bar{\bar{D}}(\alpha_i - f'_i) + \frac{1}{n^2} \bar{\bar{G}}\alpha'_i]$$

$$= - \frac{1}{n^2}[\frac{\bar{A}}{2} f''(\eta_i) + \lambda\bar{D}f''(\zeta_i) + \bar{G}\, \alpha'_{i+1}]$$

$$- \frac{1}{n^2}[\frac{\bar{\bar{A}}}{2} f''(\theta_i) + (1-\lambda)\bar{\bar{D}}f''(\sigma_i) + \bar{\bar{G}}\alpha'_i] \qquad (3.3.4)$$

$$\frac{i}{n} < \eta_i\,,\ \xi_i < \frac{i+1}{n}\,, \qquad \frac{i-1}{n} < \theta_i, \sigma_i < \frac{i}{n}\ .$$

Now setting $B_i = M_i - f''_i$, $(1 \le i \le n)$ and using (3.3.4), since $\frac{A}{2} + \lambda D + E + F + G = 0$ we have the following system of equations:

$$\bar{E}\, B_{i+1} + (\bar{F} + \bar{\bar{F}})B_i + \bar{\bar{E}}\, B_{i-1}$$

$$= - [\frac{\bar{A}}{2} (f''(\eta_i) - f''_i) + \lambda\bar{D}(f''(\zeta_i) - f''_i) + \bar{G}(\alpha'_{i+1} - f''_i)]$$

$$- [\frac{\bar{\bar{A}}}{2} (f''(\theta_i) - f''_i) + (1-\lambda)\bar{\bar{D}}(f''(\sigma_i) - f''_i) + \bar{\bar{G}}(\alpha'_i - f''_i)]$$

$$- \bar{E}(f''_{i+1} - f''_i) - \bar{\bar{E}}(f''_{i-1} - f''_i)\ . \qquad (3.3.5)$$

Since $\bar{E} + \bar{F}$ and $\bar{\bar{E}} + \bar{\bar{F}}$ are always greater than zero when $0 < \lambda < 1$, the method of [26] can be used to find an upper bound for $\mathrm{Max}_i\ |B_i|$. For if $\mathrm{Max}\ |B_i| = |B_j|$ then by the Darbous property

$$(\bar{E} + \bar{\bar{E}} + \bar{F} + \bar{\bar{F}})\ |B_j| \le K_o(\lambda)\ \omega(f''; \frac{1}{n})\ .$$

Since $\bar{E} + \bar{F}$, $\bar{\bar{E}} + \bar{\bar{F}}$ are positive numbers depending only on λ it it follows that

$$\text{Max}_i \ |B_i| \leq K_1(\lambda)\ \omega(f''; \tfrac{1}{n}) \ , \tag{3.3.6}$$

where $K_1(\lambda)$ is independent of the choice of joints. This proves that as $\frac{1}{n} \to 0$, the difference $S_i'' - f_i''$ tends to zero uniformly at all joints.

It remains to be proved that $S''(x) - f''(x)$ also approach 0 as n tends to ∞. Now, for any quintic polynomial $q(x)$ in [a,b], the following identity is valid:

$$\begin{aligned} &Aq(b) + Bq(a) + Chq'(a) + Dhq'(a+\lambda h) + Eh^2q''(a) \\ &\qquad + Fh^2q''(a+\lambda h) + Gh^5q^{(5)}(x) = 0 \ . \end{aligned} \tag{3.3.7}$$

Where

$A = 12\lambda^4$; $\qquad B = -A$;

$C = -12\lambda^3 - 12\lambda + 6$; $\qquad D = -(A+C)$;

$E = (-6\lambda^2 + 8\lambda - 9)\lambda^2$; $\qquad F = -\frac{1}{2}(A + 2D\lambda + 2E)$;

$G = -\frac{1}{30}(5\lambda^2 - 3)\lambda^4$.

Using (3.3.7) with $S(x)$ for $q(x)$, $a = \frac{i-1}{n}$, $b = \frac{i}{n}$ then we have

$$\begin{aligned} \left|G \frac{1}{n^5} S^{(5)}(x)\right| \leq \Big| &A(f_i - f_{i-1} - \tfrac{1}{n} f'_{i-1}) + \tfrac{D}{n}(\alpha_i - f'_{i-1}) \\ &+ \frac{E}{n^2} S''_{i-1} + \frac{E}{n^2} \alpha'_i \Big| \end{aligned}$$

$$\begin{aligned} \left|G \frac{1}{n^3} S^{(5)}(x)\right| &\leq \left|\tfrac{A}{2} f''(\eta_i) + D\lambda f''(\zeta_i) + Ef''_{i-1} + Ff''_i\right| \\ &\qquad + E\left|S''_{i-1} - f''_{i-1}\right| + F\left|\alpha'_i - f''_i\right| \\ &\leq \left|\tfrac{A}{2}\right| \left|f''(\eta_i) - f''_i\right| + |D\lambda| \left|f''(\zeta_i) - f''_i\right| \\ &\qquad + |E| \left|f''_{i-1} - f''_i\right| + E\left|S''_{i-1} - f''_{i-1}\right| + F\left|\alpha'_i - f''_i\right| \end{aligned}$$

$$\left|\frac{S^{(5)}(x)}{n^3}\right| \leq K_2(\lambda)\ \omega(f''; \tfrac{1}{n}) \quad . \tag{3.3.8}$$

Now $S''(x)$ is piecewise cubic $\varepsilon\ C[0,1]$ so that for $\frac{i-1}{n} < x < \frac{i}{n}$, we may write

$$S''(x) = \Lambda_i(x) + 20\gamma_i(x-x_i)(x-x_{i-1})(x-\xi_i) \tag{3.3.9}$$

where

$$\Lambda_i(x) = M_i \frac{(x-x_{i-1})(x-\xi_i)}{(x_i-x_{i-1})(x_i-\xi_i)} + M_{i-1} \frac{(x_i-x)(\xi_i-x)}{(x_i-x_{i-1})(\xi_i-x_{i-1})}$$

$$+ S''(\xi_i) \frac{(x_i-x)(x-x_{i-1})}{(x_i-\xi_i)(\xi_i-x_{i-1})}$$

$$\gamma_i = S^{(5)}(x) \quad , \qquad \frac{i-1}{n} < x < \frac{i}{n} \quad .$$

Let $\frac{i-1}{n} < x < \frac{i}{n}$ for some i. Set

$$\Lambda_i^*(x) = f''_i \frac{(x-x_{i-1})(x-\xi_i)}{(x_i-x_{i-1})(x_i-\xi_i)} + f''_{i-1} \frac{(x_i-x)(\xi_i-x)}{(x_i-x_{i-1})(\xi_i-x_{i-1})}$$

$$+ f''(\xi_i) \frac{(x_i-x)(x-x_{i-1})}{(x_i-\xi_i)(\xi_i-x_{i-1})} \quad .$$

Then it is easy to see from elementary considerations and from (3.3.8)

$$|S''(x)-f''(x)| \leq |S''(x)-\Lambda_i(x)| + |\Lambda_i(x)-\Lambda_i^*(x)| + |\Lambda_i^*(x)-f''(x)|$$

$$\leq |S''(x)-\Lambda_i(x)| + K_1(\lambda)\omega(f''; \tfrac{1}{n}) + \omega(f''; \tfrac{1}{n}) \quad . \tag{3.3.10}$$

By using (3.3.9) we obtain

$$|S''(x)-\Lambda_i(x)| \leq 20|\gamma_i| \; |(x-x_i)(x_i-x_{i-1})(x-\xi_i)|$$

$$\leq \frac{5}{2}\frac{|\gamma_i|}{n^3} \leq \frac{5}{2} K_2(\lambda)\, \omega(f''; \frac{1}{n}) \quad . \qquad (3.3.11)$$

Combining this inequality with (3.3.9), we obtain for all x

$$|S''(x)-f''(x)| \leq (\frac{7}{2} K_2(\lambda)+1)\, \omega(f''; \frac{1}{n}) \equiv K(\lambda)\, \omega(f''; \frac{1}{n}) \quad .$$

This proves Theorem III.3.1 with $r = 2$. The proof for $r = 0,1$ runs exactly as in Theorem III.1.6, and will be omitted.

Theorem III.3.2

Let $f(x) \in C^1(-\infty,\infty)$ having period 1. Let $\Delta : 0 = x_o < x_1 < \ldots < x_n = 1$ and let $S(x) \in C^1(-\infty,\infty)$ be the quintic spline on Δ with period 1, satisfying:

$$S(\xi_i) = f(\xi_i) = \alpha_i \quad , \qquad S(\eta_i) = f(\eta_i) = \beta_i \quad ,$$

$$S'(\xi_i) = f'(\xi_i) = \alpha_i' \quad , \qquad S'(\eta_i) = f'(\eta_i) = \beta_i' \quad ,$$

where

$$\xi_i = mx_i + (1-m)x_{i-1} \quad , \qquad \eta_i = \ell x_i + (1-\ell)x_{i-1} \quad ,$$

$$0 < m < \frac{1}{2} < \ell < 1 \quad , \qquad m + \ell = 1 \quad ,$$

Then for all x we have

$$|f^{(r)}(x) - S^{(r)}(x)| \leq C\, ||\Delta||^{1-r}\, \omega(f';||\Delta||) \quad , \qquad r = 0,1 \quad ,$$

where $||\Delta|| = \max_i (x_i-x_{i-1})$, C an absolute constant, which depends on ℓ , m only.

For the proof of this theorem we shall need the following:

Lemma

If $P(x)$ is any quintic polynomial in $[a,b]$ then the following identity is valid.

$$AP(\xi) + BP(\eta) - (A+B)P(a) + ChP'(\xi) + DhP'(\eta) + EhP'(b) + FhP'(a) = 0$$

where $\quad \xi = a + m(b-a), \quad \eta = a + \ell(b-a), \quad h = b-a \quad .$

$$A = 5P_1P_3\ell^4 - 40P_1P_5\ell^2 - 5P_2P_4\ell^4$$

$$B = 5P_2P_4m^4 + 40P_2P_6m^2 - 5P_1P_3m^4$$

$$C = -P_1P_3\ell^4m + 8P_1P_5\ell^2m + P_2P_4\ell^4m + 2P_1\ell^3m(\ell-m)$$

$$D = P_1P_3m^4\ell - 8P_2P_6m^2\ell - P_2P_4m^4\ell - 2P_2m^3\ell(\ell-m)$$

$$E = 2m^3\ell^3(\ell-m)(P_1M^2 - P_2\ell^2)$$

$$F = -(Am+B\ell + C + D + E)$$

and

$$P_1 = (\ell-1)(9\ell^2-15\ell+m\ell+25m-8\ell^2m)$$

$$P_2 = (m-1)(9m^2-15m+m\ell+25\ell-8m^2\ell)$$

$$P_3 = 3m^2 - 8m + 5$$

$$P_4 = 3\ell^2 - 8\ell + 5$$

$$P_5 = \ell(\ell-m)(1-m)$$

$$P_6 = m(\ell-m)(1-\ell)$$

Proof of Theorem III.3.2: For the sake of brevity, set $N_i = S'(x_i)$, $S_i = S(x_i)$ and $h_i = x_i-x_{i-1}$, $i = 1,2,\ldots,n$. Then using the lemma first with $a = x_i$, $b = x_{i+1}$, $\xi = \xi_{i+1}$, $\eta = \eta_{i+1}$, next with $a = x_i$, $b = x_{i-1}$, $\xi = \eta_i$, $\eta = \xi_i$ and eliminating S_i from the two equations so obtained, we have for $i = 1,2,\ldots,n$.

$$A\alpha_{i+1} - A\beta_i + B\beta_{i+1} - B\alpha_i + Ch_{i+1}\alpha'_{i+1} + Ch_i\beta'_i$$
$$+ Dh_{i+1}\beta'_{i+1} + Dh_i\alpha'_i + Eh_{i+1}S'_{i+1} + Eh_iS'_{i-1}$$
$$+ Fh_{i+1}S'_i + Fh_iS'_i = 0 \quad . \qquad (3.3.16)$$

Since the spline S interpolates to f at the points ξ_i, η_i, $(i = 1,2,\ldots,n)$, we have

$$\beta_i = f(\eta_i) = f(\xi_i) + (\eta_i-\xi_i)f'(\rho_i)$$
$$= \alpha_i + (\ell-m)h_if'(\rho_i) \quad , \qquad \xi_i < \rho_i < \eta_i \ . \qquad (3.1.17)$$

Substituting (3.3.17) into (3.3.16) to replace β_{i+1} and α_i we have

$$(A+B)(\alpha_{i+1}-\beta_i) + B(\ell-m)(h_{i+1}f'(\rho_{i+1}) + h_if'(\rho_i))$$
$$+ C(h_{i+1}\alpha'_{i+1}+h_i\beta'_i) + D(h_{i+1}\beta'_{i+1}+h_i\alpha'_i)$$
$$= -Eh_{i+1}N_{i+1} - (Fh_{i+1}+Fh_i)N_i - Eh_iN_{i-1} \ ,$$
$$(i = 1,2,\ldots,n) \quad . \qquad (3.3.18)$$

After the application of the mean value theorem and the Darboux property of a derivative the left hand side of (3.3.18) becomes

$$m(A+B)(h_{i+1}+h_i)f'(\tau_i) + B(\ell-m)(h_{i+1}+h_i)f'(\sigma_i)$$
$$+ C(h_{i+1}\alpha'_{i+1}+h_i\beta'_i) + D(h_{i+1}\beta'_{i+1}+h_i\alpha'_i) \qquad (3.3.19)$$

where $x_{i-1} < \tau_i < x_{i+1}$ and $x_{i-1} < \sigma_i < x_{i+1}$.

Setting $B_i = N_i-f'_i$, $(1 \le i \le n)$ and using (3.3.19) we have, from (3.3.18), the following system of equations:

$$-Eh_{i+1}B_{i+1} - F(h_{i+1}+h_i)B_i - Eh_iB_{i-1}$$

$$= (A+B)m(h_{i+1}+h_i)(f'(\tau_i)-f'_i) + B(\ell-m)(h_{i+1}+h_i)(f'(\sigma_i)-f'_i)$$

$$+ Ch_{i+1}(\alpha'_{i-1}-f'_i) + Ch_i(\beta_i-f'_i) + Dh_{i+1}(\beta'_{i+1}-f'_i)$$

$$+ Dh_i(\alpha'_i-f'_i) + Eh_{i+1}(f'_{i+1}-f'_i) + Eh_i(f'_{i-1}-f'_i) \quad .$$

Since $-E-F = 14\,\ell(1-\ell)(2\ell-1)(5-4\ell)(4\ell+1)$ always greater than zero the method of [26] can be used to find an upper bound for $\max_i |B_i|$. For if $\max_i |B_i| = |B_j|$ then

$$-(E+F)(h_{j+1}+h_j)\ |B_j| \leq C_o(\ell,m)(h_{j+1}+h_j)\ \omega(f';||\Delta||) \ .$$

It follows that

$$\max_i |B_i| \leq C_1(\ell,m)\ \omega(f';||\Delta||) \quad .$$

This proves that as $||\Delta|| \to 0$, the difference $S'_i - f'_i$ tends to zero uniformly at all joints.

It remains to be proved that $S'(x) - f'(x)$ also approaches zero as $||\Delta||$ tends to zero. Now for $x_{i-1} \leq x \leq x_i$,

$$S(x) = \lambda_i(x) + \psi_i(x) \quad ,$$

where $y = \lambda_i(x)$ is the straight line through the points (ξ_i,α_i) and (η_i,β_i). Since

$$\psi_i(x) = \gamma_i(x-\xi_i)(x-\eta_i)(x^3+a_ix^2+b_ix+c_i) \quad ,$$

with suitable γ_i, a_i, b_i and c_i we have

$$N_i = \lambda_i' + \gamma_i[h_i(x_i^3+a_ix_i^2+b_ix_i+c_i) + \ell m h_i^2(3x_i^2+2a_ix_i+b_i)]$$

$$N_{i-1} = \lambda_i' + \gamma_i[-h_i(x_{i-1}^3+a_ix_{i-1}^2+b_ix_{i-1}+c_i) + \ell m h_i^2(3x_{i-1}^2+2a_ix_{i-1}+b_i)]$$

$$S'(\eta_i) = \lambda_i' + \gamma_i(\ell-m)h_i(\eta_i^3+a_i\eta_i^2+b_i\eta_i+c_i)$$

$$S'(\xi_i) = \lambda_i' + \gamma_i(m-\ell)h_i(\xi_i^3+a_i\xi_i^2+b_i\xi_i+c_i) \quad . \tag{3.3.20}$$

Hence

$$(\ell-m)^2(N_i+N_{i-1}) - (1+2\ell m)(S'(\eta_i)+S'(\xi_i))$$
$$= -12\ell m\lambda_i + 2\gamma_i\ell m(1+m\ell)(\ell-m)^2\, h_i^4 \quad , \tag{3.3.21}$$

also from the definition of the N_i's, it follows that

$$S'(x) = \Lambda_i(x) + 5\gamma_i(x-x_i)(x-x_{i-1})(x-\eta_i)(x-\xi_i)$$

with

$$\Lambda_i(x) = (N_{i-1}\frac{x_i-x}{h_i} + N_i\frac{x-x_{i-1}}{h_i})(x-\eta_i)(x-\xi_i)\frac{1}{\ell m h^2}$$
$$- (\alpha_i'\frac{\eta_i-x}{\eta_i-\xi_i} + \beta_i'\frac{x-\xi_i}{\eta_i-\xi_i})(x-x_i)(x-x_{i-1})\frac{1}{\ell m h^2} \quad .$$

Thus

$$|S'(x)-\Lambda_i(x)| \le 5|\gamma_i|\ |(x-x_i)(x-x_{i-1})(x-\eta_i)(x-\xi_i)| \le \frac{5}{16}|\gamma_i|\ h_i^4 \ ,$$

so that using (3.3.21) we have

$$|S'(x)-\Lambda_i(x)|$$
$$\le \frac{5(\ell-m)^2(N_i+N_{i-1})-(1+2\ell m)(S'(\eta_i)+S'(\xi_i))+(2\ell m\lambda_i)}{32\ \ m(1+m\ell)(\ell-m)^2} \ , \tag{3.3.22}$$

since

$$\lambda_i'(x) = \frac{\beta_i-\alpha_i}{\eta_i-\xi_i} = f'(\theta_i) \quad , \qquad \xi_i < \theta_i < \eta_i \; .$$

The right hand side of (3.3.22) is

$$\le \frac{5}{32\ell m(1+m\ell)(\ell-m)^2}[(\ell-m)^2(|B_i|+|B_{i-1}|)+(1+2m\ell)$$

$$\times (|f_i'-S'(\eta_i)|+|f_{i-1}'-S'(\xi_i)|)$$

$$+ 6m\ell(|f'(\theta_i)-S'(\eta_i)|+|f'(\theta_i)-S'(\xi_i)|)$$

$$\le \frac{5}{32\ell m(1+m\ell)(\ell-m)^2}[2(\ell-m)^2C_1(\ell,m)\omega(f';||\Delta||)+(2+16m\ell)\omega(f';||\Delta||)]$$

$$= C_2(\ell,m)\omega(f';||\Delta||) \; .$$

Define

$$\Lambda_i^*(x) = (f_{i-1}' \frac{x_i-x}{h_i} + f_i' \frac{x-x_{i-1}}{h_i})(x-\eta_i)(x-\xi_i) \frac{1}{m\ell h^2}$$

$$- (f'(\xi_i) \frac{\eta_i-x}{\eta_i-\xi_i} + f'(\eta_i) \frac{x-\xi_i}{\eta_i-\xi_i})(x-x_i)(x-x_{i-1}) \frac{1}{m\ell h^2}$$

observe that $|\Lambda_i-\Lambda_i^*| \le \max_i |B_i|$ for all $x \in [x_{i-1},x_i]$ and that $|\Lambda_i^*-f'(x)| \le \frac{1}{2}\,\omega(f';||\Delta||)$.

Hence we have

$$|S'(x)-f'(x)| \le |S'(x)-\Lambda_i(x)| + |\Lambda_i(x) - \Lambda_i^*(x)| + |\Lambda_i^*(x) - f'(x)|$$

$$\le (C_2(\ell,m) + C_1(\ell,m) + \tfrac{1}{2})\,\omega(f';||\Delta||)$$

$$= C(\ell,m)\;\omega(f';||\Delta||) \quad . \qquad (3.3.23)$$

where C is an absolute constant independent of Δ . By (3.3.23) and Rolle's theorem we obtain, by intergration,

$$|S(x) - f(x)| \leq C(\ell,m)\ ||\Delta||\ \omega(f';||\Delta||) \quad .$$

Thus the proof of Theorem III.3.2 is complete.

REMARKS

Let C denote the Banach space of all continuous, real-valued functions f on $[0,1]$ such that $f(0) = f(1)$. The norm in C is $||f|| = \max\{|f(x) : 0 \leq x \leq 1\}$. To each division of the interval into n subintervals $\{\Delta : 0 = x_o < \ldots < x_n = 1\}$ there corresponds a subspace $S = S(\Delta)$ in C whose elements are the periodic cubic splines having nodes at $x_o, \ldots, x_n$. The dimension of S is n. For each $f \varepsilon C$ there is a unique element $s \varepsilon S$ which interpolates to f at the nodes: $f(x_i) = S(x_i)$ for $i = 1,2,\ldots,n$. The mapping $L : f \to S$ thus defined is a linear projection of C onto S.

Now consider a sequence of such nodal arrays $\{\Delta_n : 0 = x_{n,1} < x_{n,2} < \ldots < x_{n,n} = 1\}_{n=1}^{\infty}$. There corresponds a sequence of subspaces $S(\Delta_n)$ and a sequence of projection L_n. Cheney and Schurer [9] raise the following three open problems:

<u>Question 1</u>. Is there a linear projection A of C onto S such that $||f - Af|| \leq C\omega(f;||\Delta||)$?

<u>Question 2</u>. What is the linear projection of minimum norm from C onto S? Is it unique?

<u>Question 3</u>. What conditions on the nodes are equivalent to the inequality $\sup_n ||L_n|| < \infty$?

The discussion of Question 3 is contained in §II.6. The results are: $P_n \leq P < 2.439^+$ (for all n) is a sufficient condition that $\sup_n ||L_n|| < \infty$. If $P > 2.62^-$ there exists a sequence $\{\Delta_n\}$ such that $\lim_n \sup_n ||L_n|| = \infty$.

If we use the quintic spline to replace the cubic spline in S then the same three questions can be raised. For the analogous results of §II.6 we use a remark of Marsden [16] to obtain the existence of an upper bound P such that $P_n \leq P'$ implies $\sup_n ||L_n|| < \infty$ and if $P > 5.6^+$ then there exists a sequence of $\{\Delta_n\}$ such that $\lim_n \sup_n ||L_n|| = \infty$.

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